Method of intervals for the study of diffusion-limited annihilation, $A+A\rightarrow 0$

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We introduce a method of intervals for the analysis of diffusion-limited annihilation, $A+A\rightarrow 0$, on the line. The method leads to manageable diffusion equations whose interpretation is intuitively clear. As an example, we treat the following cases: (a) annihilation in the infinite line and in infinite (discrete) chains; (b) annihilation with input of single particles, adjacent particle pairs, and particle pairs separated by a given distance; (c) annihilation, $A+A\rightarrow 0$, along with the birth reaction $A\rightarrow 3A$, on finite rings, with and without diffusion.

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I. INTRODUCTION

Diffusion-limited reactions display a wide range of behavior characteristic of nonequilibrium dynamics, such as self-organized criticality, pattern formation, and dynamic phase transitions. However, the exact description of this behavior can only be determined for the simplest reaction schemes. Single-species annihilation, $A+A\rightarrow 0$, and coagulation, $A+A\rightarrow A$, are among the few examples of exactly solvable diffusion-limited reactions. The method of empty intervals has provided many exact results concerning diffusion-limited coagulation [1–5]. Its advantage lies in the fact that it leads to simple diffusion equations that are easy to solve, and whose interpretation is intuitively obvious. We develop a similar method of intervals for the study of diffusion-limited annihilation reactions.

Diffusion-limited annihilation has been studied in a variety of forms. For the basic annihilation process, the exact time-dependent concentration of particles has been determined for various initial distributions on the infinite one-dimensional lattice and the continuous real line [6–16]. The inclusion of particle input in the reaction scheme allows for a nontrivial steady-state concentration. Both input of adjacent particle pairs [17,18] and single-particle input [17,19,20] have been studied. A nontrivial steady state may also occur when a birth reaction is included. Annihilation with the symmetric birth reaction, $A\rightarrow 3A$, has been studied for its implications to interacting particle systems and universality theories [21–28]. Sudbury [23] referred to this reaction as the double branching annihilating random walk (DBARW). Without diffusion, the process is known as the double branching annihilating process (DBAP). We show how a method of intervals, previously introduced in the literature [24,29–32], may be used to model all of the above reactions. This powerful yet simple method yields insights that extend beyond the known results [33].

The rest of this paper is organized as follows. In Sec. II, we introduce the method of intervals, as adapted for the annihilation process $A+A\rightarrow 0$, and reproduce its well-known kinetics. Section III deals with various cases of input. Input of particle pairs separated by a distance y is considered, and the limiting cases of input of single particles ($y\rightarrow\infty$) and adjacent particle pairs ($y\rightarrow 0$) are examined. We also discuss input at a rate proportional to a function of the global concentration of particles. In Sec. IV, we present our results for DBARW and DBAP. We conclude in Sec. V with a discussion and open directions of research suggested by the new method of intervals.

II. THE METHOD OF ODD/EVEN INTERVALS

We now adapt the method of empty intervals, conceived originally for the analysis of diffusion-limited coalescence, $A+A\rightarrow A$, to reactions such as annihilation, $A+A\rightarrow 0$, where the parity (the number of particles modulo 2) is conserved. Consider the process $A+A\rightarrow 0$, taking place in a one-dimensional lattice. The particles hop randomly to the nearest site on their right or left, at equal rate Γ, and annihilate immediately upon encounter. Let $G_n(t)$ be the probability that an arbitrary segment of n consecutive sites contains an even number of particles, at time t. (We assume, for the moment, that the system is infinite and homogeneous.) A site can be either empty or occupied by a single particle, so the probability that a site is occupied, i.e., the particle density, is

$$\rho(t)=1-G_1(t). \tag{1}$$

Since the reaction $A+A\rightarrow 0$ conserves parity, the only way that G_n might change is when particles at the edge of the segment hop outside, or when particles just outside of the segment hop inside. To describe these events, we require $F_n(t)$, which is the probability that an n segment containing an even number of particles is followed by the presence of a particle at the $(n+1)$th site. This can be expressed in terms of G_n (Fig. 1):

$$2F_n(t)=(1-G_1)+(G_n-G_{n+1}). \tag{2}$$

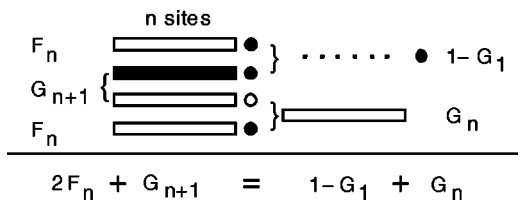

FIG. 1. Computation of $F_n(t)$: Empty (solid) rectangles symbolize n segments with an even (odd) number of particles. Empty (solid) circles represent empty (occupied) sites.

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Likewise, $H_n(t)$, the probability that an n segment containing an *odd* number of particles is followed by a particle at the $(n+1)$th site, is

$$2H_n(t)=(1-G_1)-(G_n-G_{n+1}). \tag{3}$$

The evolution equation for G_n is now readily obtained:

$$\frac{\partial}{\partial t}G_n(t)=2\Gamma(F_{n-1}-H_{n-1}+H_n-F_n). \tag{4}$$

The term proportional to F_{n-1} on the right-hand side expresses the event that a particle at site n jumps out of the segment, leaving an even number of particles in the remaining $n-1$ sites (and hence in the n segment); H_{n-1} corresponds to the same case, but when there are initially an even number of particles in the n segment [that is, an odd number in the $(n-1)$ segment]; H_n and F_n pertain to a particle just outside of the n segment, at site $n+1$, jumping in. The factor of 2 accounts for events taking place at *both* edges of the segment, at equal rate. Using Eqs. (2) and (3), this becomes

$$\frac{\partial}{\partial t}G_n(t)=2\Gamma(G_{n-1}-2G_n+G_{n+1}). \tag{5}$$

The case of $n=1$ requires a special equation, since G_0 is undefined. Taking into account all the ways G_1 might change, one finds

$$\frac{\partial}{\partial t}G_1(t)=2\Gamma(1-2G_1+G_2). \tag{6}$$

Thus, Eq. (5) may be understood to be valid for all $n\geqslant 1$, provided that one uses the boundary condition

$$G_0(t)=1. \tag{7}$$

Additionally, since the G_n are *probabilities*, we have

$$0\leqslant G_n(t)\leqslant 1. \tag{8}$$

The initial distribution of particles determines the required initial condition, $G_n(0)$. For example, suppose that at the start of the process each site is occupied randomly and independently of other sites, with probability ρ_0. Then

$$G_n(0)=\tfrac{1}{2}+\tfrac{1}{2}(1-2\rho_0)^n. \tag{9}$$

Equation (5) may be solved by standard techniques, for example by Laplace-transforming with respect to time, fitting an exponential solution to the resulting difference equation, and finally inverting the Laplace-transformed solution [34]. With the boundary conditions (7) and (8), and the natural initial condition (9), one obtains

$$G_n(\tau)=\tfrac{1}{2}+\tfrac{1}{2}(1-2\rho_0)^n e^{\beta\tau} + \frac{n}{2}\int_0^\tau e^{-2\tau'}(1-e^{\beta\tau-\beta\tau'})I_n(2\tau')\frac{d\tau'}{\tau'}, \tag{10}$$

where $\tau=2\Gamma t$, $\beta=4\rho_0^2/(1-2\rho_0)$, and $I_n(\,)$ is the modified Bessel function of order n [35]. In particular, the probability that a site contains a particle is

$$\rho(\tau)=1-G_1(\tau) =\tfrac{1}{2}-\tfrac{1}{2}(1-2\rho_0)e^{\beta\tau} -\tfrac{1}{2}\int_0^\tau e^{-2\tau'}(1-e^{\beta\tau-\beta\tau'})I_1(2\tau')\frac{d\tau'}{\tau'}. \tag{11}$$

It is often more convenient to assume a low initial density of particles, and work in the continuum limit. Let a be the lattice spacing. Then, setting $x=na$, $G_n(t)\to G(x,t)$, and $\Gamma=D/a^2$ in Eq. (5), and taking the limit $a\to 0$, one obtains the diffusion equation

$$\frac{\partial}{\partial t}G(x,t)=2D\frac{\partial^2}{\partial x^2}G(x,t), \tag{12}$$

with the boundary conditions

$$G(0,t)=1, \tag{13a}$$

$$0\leqslant G(x,t)\leqslant 1. \tag{13b}$$

The particle concentration is obtained from

$$c(t)=\lim_{a\to 0}\frac{\rho(t)}{a}=-\frac{\partial}{\partial x}G(x,t)\big|_{x=0}. \tag{14}$$

Consider, for example, an initial concentration c_0 of randomly placed particles. The initial condition is obtained from Eq. (9), setting $\rho_0=c_0a$ and passing to the continuum limit:

$$G(x,0)=\tfrac{1}{2}+\tfrac{1}{2}e^{-2c_0x}. \tag{15}$$

Solving for the concentration is then straightforward:

$$c(t)=c_0e^{z^2}\mathrm{erfc}(z),\quad z=2c_0\sqrt{2Dt}. \tag{16}$$

This is the known result, with its familiar long-time asymptotic behavior,

$$c(t)\sim\frac{1}{\sqrt{8\pi Dt}},\quad t\to\infty. \tag{17}$$

III. ANNIHILATION WITH INPUT

Let us now include the possibility of input, where empty sites become spontaneously occupied at a prescribed rate. We shall assume that the input of particles is homogeneous (translation-invariant). Various cases of input may be analyzed through the method of intervals within this restriction.

A. Input of single particles

Consider a random, homogeneous input of single particles at constant rate R per unit space per unit time. If the input site is already occupied, we assume that the particles react immediately and the site becomes empty. In other words, the

state of individual sites flips (from empty to occupied, and vice versa) at rate Ra per unit time. This kind of input affects the rate of change of G_n, thus

$$\left(\frac{\partial G_n}{\partial t}\right)_{A\ \text{input}} = nRa(1-G_n) - nRaG_n = nRa(1-2G_n). \tag{18}$$

The term $nRa(1-G_n)$ accounts for the increase in G_n due to the input of a particle to an n-site interval, initially containing an *odd* number of particles, while $-nRaG_n$ is the (negative) change in G_n when a particle is input into an initially *even* n interval. Note that Eq. (18) is valid for all $n \geqslant 1$, and so the input does not affect the boundary condition (8). In the continuum limit, the change due to input is $\partial G/\partial t = Rx(1-2G)$, so Eq. (12) becomes

$$\frac{\partial}{\partial t}G(x,t) = 2D\frac{\partial^2}{\partial x^2}G(x,t) + Rx[1-2G(x,t)], \tag{19}$$

subject to the same boundary conditions as without input. The problem can be solved by substituting

$$K(x,t) = 1-2G(x,t).$$

$K(x,t)$ then satisfies

$$\frac{\partial}{\partial t}K(x,t) = 2D\frac{\partial^2}{\partial x^2}K(x,t) - 2RxK(x,t),$$

$$K(0,t) = -1.$$

Using the method of separation of variables, we write

$$K(x,t) = \sum_{\lambda \geqslant 0} a_\lambda K_\lambda(x) e^{-\lambda t},$$

where the eigenfunctions satisfy

$$2D\frac{\partial^2}{\partial x^2}K_\lambda(x) = (2Rx-\lambda)K_\lambda(x).$$

Thus,

$$K_\lambda(x) = \mathrm{Ai}\left(\left(\frac{R}{D}\right)^{1/3}x - \frac{\lambda}{2(DR^2)^{1/3}}\right),$$

where Ai( ) is the Airy function, a solution to the equation $\mathrm{Ai}''(z) = z\,\mathrm{Ai}(z)$ [35]. The boundary condition $K(0,t) = -1$ implies $K_0(0) = -1$ and $K_\lambda(0) = 0$ $(\lambda > 0)$. The condition $K_0(0) = -1$ yields the steady-state solution,

$$G_s(x) = \frac{1}{2} + \frac{\mathrm{Ai}((R/D)^{1/3}x)}{2\,\mathrm{Ai}(0)}, \tag{20}$$

and the steady-state concentration,

$$c_s = -\frac{\partial}{\partial x}G_s(x)\big|_{x=0} = \frac{|\mathrm{Ai}'(0)|}{2\,\mathrm{Ai}(0)}\left(\frac{R}{D}\right)^{1/3}. \tag{21}$$

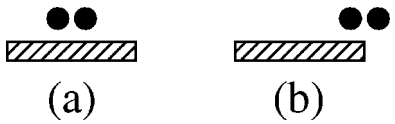

FIG. 2. Change of G_n with input of adjacent particle pairs: Hatched rectangles represent n segments with initially an even (odd) number of particles. (a) Input of a pair inside the interval does not affect the parity. (b) Input of the pair at the edge of the segment, when just one particle lands inside, shifts the segment parity to odd (even).

The condition $K_\lambda(0) = 0$ imposes discrete eigenvalues:

$$\lambda_n = 2(DR^2)^{1/3}|a_n|,$$

where a_n are the zeros of the Airy function. The relaxation time to the stationary state, τ, is then given by the minimal eigenvalue,

$$\tau^{-1} = \lambda_{\min} = 2(DR^2)^{1/3}|a_1| = 4.6762(DR^2)^{1/3}. \tag{22}$$

B. Input of adjacent particle pairs

Consider now homogeneous, random input of particle *pairs* to adjacent sites: any two adjacent sites become occupied at rate $Ra = r$ per unit time. As before, if a target site is already occupied, the site becomes empty as a result of input. The situation is analogous to that of input of single particles, but the parity of an interval is now only affected when input occurs at the interval's edge, ie., when only one particle of the pair falls right inside the interval. G_n is increased by input at the edge of an odd n interval and decreased by input at an even interval (see Fig. 2). Thus, the rate of change of G_n due to pair input is

$$\left(\frac{\partial}{\partial t}G_n\right)_{AA\ \text{input}} = 2[r(1-G_n) - rG_n] = 2r(1-2G_n). \tag{23}$$

In the continuum limit, $G(x,t)$ must satisfy

$$\frac{\partial}{\partial t}G(x,t) = 2D\frac{\partial^2}{\partial x^2}G(x,t) + 2r[1-2G(x,t)], \tag{24}$$

with the usual boundary condition $G(0,t) = 1$.

Again, we use $K(x,t) = 1-2G(x,t)$ to determine the solution. $K(x,t)$ must satisfy

$$\frac{\partial}{\partial t}K(x,t) = 2D\frac{\partial^2}{\partial x^2}K(x,t) - 4rK(x,t),$$

$$K(0,t) = -1.$$

Expanding $K(x,t) = \Sigma_\lambda K_\lambda(x)e^{-\lambda t}$, as before, we find

$$2D\frac{\partial^2}{\partial x^2}K_\lambda(x) = (4r-\lambda)K_\lambda(x),$$

with $K_0(0)=-1$ and $K_\lambda(0)=0$ $(\lambda>0)$. This leads to the stationary solution

$$G_s(x)=\tfrac{1}{2}+\tfrac{1}{2}e^{-\sqrt{(2r/D)}x} \tag{25}$$

and the stationary concentration

$$c_s=\frac{1}{2}\left(\frac{2r}{D}\right)^{1/2}. \tag{26}$$

The relaxation spectrum is continuous:

$$\lambda>4r; \quad K_\lambda(x)=\sin\left(\sqrt{\frac{\lambda-4r}{2D}}x\right), \tag{27}$$

and the relaxation time is

$$\tau^{-1}=\lambda_{\min}=4r. \tag{28}$$

These results, along with the single-particle input results, are in complete agreement with previous work by Rácz [17] and Lushnikov [18].

C. Input of correlated pairs

The method of intervals allows us to deal with more complicated input. As an example, we consider input of particle pairs, separated by m lattice spacings. This input of *correlated* pairs interpolates between the two cases discussed so far: for $m=0$, the particle pairs are adjacent, just as in Sec. III B, while the case of Sec. III A is recovered as $m\to\infty$, since then the input particles cannot possibly affect each other and the correlation is lost.

Suppose that the particle pairs are deposited at rate Ra per unit time (per site). The rate of change of G_n due to input is now different for $n\leqslant m$ and $n>m$:

$$\left(\frac{\partial}{\partial t}G_n\right)_{A\cdots A \text{ input}}=2Rna(1-2G_n), \quad n\leqslant m, \tag{29a}$$

$$\left(\frac{\partial}{\partial t}G_n\right)_{A\cdots A \text{ input}}=2R(m+1)a(1-2G_n), \quad n>m. \tag{29b}$$

Again, Eq. (29a) does not affect the usual boundary condition $G_0=1$. Adding these contributions to Eq. (5) and taking the continuum limit, we obtain

$$\frac{\partial}{\partial t}G(x,t)=2D\frac{\partial^2}{\partial x^2}G(x,t)+2Rx[1-2G(x,t)], \quad x\leqslant y, \tag{30a}$$

$$\frac{\partial}{\partial t}G(x,t)=2D\frac{\partial^2}{\partial x^2}G(x,t)+2Ry[1-2G(x,t)], \quad x>y, \tag{30b}$$

with the usual boundary condition $G(0,t)=1$. Additionally, $G(x,t)$ and $\partial G(x,t)/\partial x$ are continuous at $x=y$.

For simplicity, we analyze only the steady state. We find

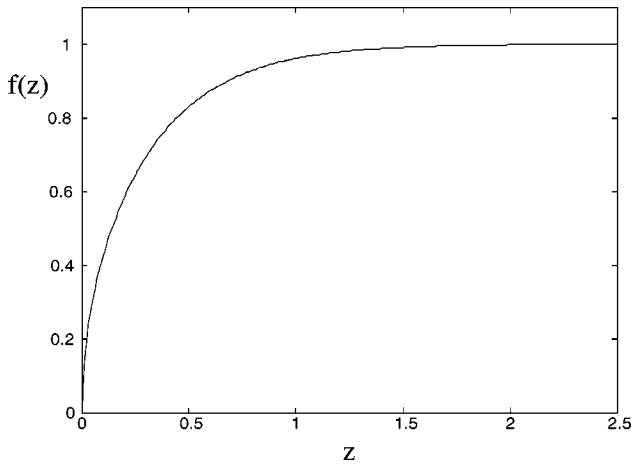

FIG. 3. The crossover function $f(z)$ of Eq. (33). Notice the limiting behavior for small and large z and the crossover about $z\approx 1$ evident from the plot.

$$G_s(x)=c_1\mathrm{Ai}(\alpha x)+c_2\mathrm{Bi}(\alpha x)+\tfrac{1}{2}, \quad x\leqslant y, \tag{31a}$$

$$G_s(x)=c_3e^{-\sqrt{\alpha^3 y}\,x}+\tfrac{1}{2}, \quad x>y, \tag{31b}$$

where Bi( ) is the independent, divergent solution to Airy's equation [35], $\alpha=(2R/D)^{1/3}$, and c_1,c_2,c_3 are constants obtained from the boundary conditions at $x=0$ and $x=y$. The steady-state concentration is

$$c_s=\frac{|\mathrm{Ai}'(0)|}{2\,\mathrm{Ai}(0)}\left(\frac{2R}{D}\right)^{1/3}f\left[\left(\frac{2R}{D}\right)^{1/3}y\right], \tag{32}$$

where

$$f(z)=\frac{\sqrt{z}\mathrm{Bi}(z)+\mathrm{Bi}'(z)+\sqrt{3z}\mathrm{Ai}(z)+\sqrt{3}\mathrm{Ai}'(z)}{\sqrt{z}\mathrm{Bi}(z)+\mathrm{Bi}'(z)-\sqrt{3z}\mathrm{Ai}(z)-\sqrt{3}\mathrm{Ai}'(z)} \to\begin{cases}\dfrac{\sqrt{z}}{2}, & z\ll 1,\\ 1, & z\gg 1.\end{cases} \tag{33}$$

Equation (33) expresses the crossover behavior between input of single particles and adjacent particle pairs. Indeed, when $y\to\infty$ $(z\to\infty)$, the particles are essentially uncorrelated, and the steady state is the same as for single particles, of Eq. (21), but with $2R$ instead of R. This is because the input of pairs introduces particles at twice the rate of single particles. The limit of $y\to 0$ $(z\to 0)$ yields $c_s\sim\sqrt{(R/D)y}$, just as for adjacent particle pairs [Eq. (26)]. The crossover between the two regimes occurs about $(R/D)^{1/3}y\sim 1$ (Fig. 3).

D. Input proportional to some global property

We can also analyze input at a rate proportional to some global property of the system. Consider input of single particles, as in Sec. III A, but at a rate proportional to a functional of $G(x,t)$:

$$R=\mathcal{R}[G(x,t)].$$

Putting this rate into Eq. (19) yields a nonlocal partial differential equation for $G(x,t)$, which is generally difficult to solve. However, the steady state may be obtained in the following manner. In the steady-state limit, the rate $R_s=\mathcal{R}[G_s(x)]$ is constant. This constant can be found by solving

$$R_s=\mathcal{R}[G_s(x|R_s)], \tag{34}$$

where $G_s(x|R)$ is the distribution given by Eq. (20), assuming constant rate R. Once the value of R_s is known, the steady state distribution, $G_s(x|R_s)$, and steady-state concentration, c_s, follow. Notice that the same procedure applies for other kinds of input, with minor changes: for example, for input of adjacent particle pairs, $G_s(x|R)$ is given by Eq. (25) rather than Eq. (20).

An amusing example is input of single particles at a rate proportional to a power of the particle concentration:

$$R=R_0\left(\frac{c(t)}{c(0)}\right)^{\alpha}. \tag{35}$$

In this case, we can find c_s directly from Eq. (21):

$$c_s=\gamma\left(\frac{R_s}{D}\right)^{1/3}, \quad \gamma=\frac{|\mathrm{Ai}'(0)|}{2\,\mathrm{Ai}(0)},$$

or

$$c_s=\left(\frac{\gamma^3}{c(0)^{\alpha}}\frac{R_0}{D}\right)^{1/(3-\alpha)}. \tag{36}$$

The true meaning of Eq. (36), for different values of α, is revealed only from a careful analysis of the pertinent effective-rate equation [1]:

$$\frac{d}{dt}c=-k_1c^3+k_2c^{\alpha}, \quad k_1\sim D, \quad k_2\sim R_0/c(0)^{\alpha}. \tag{37}$$

Equation (37) has a steady state similar to Eq. (36), but one can clearly see that state is stable only for $\alpha<3$. For $\alpha>3$, the solution is unstable; instead, the system flows to one of the stable states $c=0$ or $c=\infty$. A particularly intriguing case, in which fluctuations not reflected in the mean-field analysis will surely play a central role, is that of $\alpha=3$.

IV. ANNIHILATION WITH SYMMETRIC BIRTH

We now treat the process of annihilation, $A+A\to 0$, with the addition of the backreaction $A\to 3A$, where particles give birth to two new particles at the adjacent sites to their left and right, at rate Ω (per particle, per unit time). If birth takes place on a site that is already occupied, annihilation is immediate and the site becomes empty. This process has been analyzed by Sudbury [23], who named it the double branching annihilating random walk (DBARW). Sudbury [23] also studied the double branching annihilating process (DBAP), which is the same as the DBARW except that the particles do not diffuse. The DBARW is now commonly referred to as the branching-annihilating walk with two offspring ($n=2$-BAW), but since we wish to distinguish between the processes with and without diffusion, we shall utilize Sudbury's nomenclature.

Consider the DBARW taking place on an N-site chain, with periodic boundary conditions (or an N-site ring). Because the ring is finite, G_N is determined by the parity of the initial number of particles, and remains constant throughout the process. The effect of diffusion on G_n has already been discussed, and is given by Eqs. (5) and (6):

$$\left(\frac{\partial}{\partial t}G_n\right)_{\mathrm{diff}}=2\Gamma(G_{n-1}-2G_n+G_{n+1}), \quad 1<n<N$$

and

$$\left(\frac{\partial}{\partial t}G_1\right)_{\mathrm{diff}}=2\Gamma(1-2G_1+G_2), \quad n=1.$$

Birth affects G_n in a similar way to diffusion: the parity of an n segment changes only when a particle just inside or just outside the segment gives birth. Thus,

$$\left(\frac{\partial}{\partial t}G_n\right)_{\mathrm{birth}}=2\Omega(G_{n-1}-2G_n+G_{n+1}), \quad 1<n<N-1.$$

For $n=1$, birth into an empty site decreases G_1, while birth into an occupied site increases G_1. The two effects add up to

$$\left(\frac{\partial}{\partial t}G_1\right)_{\mathrm{birth}}=2\Omega(G_2-G_1), \quad n=1.$$

Finally, for $n=N-1$, birth from a particle at the inner edge of the $(N-1)$ segment is similar to the case of generic n, but birth from a particle outside of the segment is different: because the lattice is an N ring, a particle outside of the $(N-1)$ segment gives birth to *two* particles inside it (one at each edge), and the parity does not change. Thus,

$$\left(\frac{\partial}{\partial t}G_{N-1}\right)_{\mathrm{birth}}=2\Omega(G_{N-2}-G_{N-1}), \quad n=N-1.$$

Putting the contributions from diffusion and from birth together, we get

$$\frac{\partial}{\partial t}G_n=2(\Gamma+\Omega)(G_{n-1}-2G_n+G_{n+1}), \quad 1<n<N-1, \tag{38a}$$

$$\frac{\partial}{\partial t}G_1=2\Gamma(1-2G_1+G_2)+2\Omega(G_2-G_1), \quad n=1, \tag{38b}$$

$$\frac{\partial}{\partial t}G_{N-1}=2\Gamma(G_{N-2}-2G_{N-1}+G_N)+2\Omega(G_{N-2}-G_{N-1}), \quad n=N-1. \tag{38c}$$

For simplicity, we focus on the steady state, where time derivatives are zero. (The transient behavior may be analyzed by methods similar to that of Sec. II.) The general (steady-state) solution of Eq. (38a) is $G_n = An + B$, where A and B are constants. Their explicit value is determined from the boundary conditions (38b) and (38c):

$$\Omega A - \Gamma B = -\Gamma, \tag{39a}$$

$$(N\Gamma + \Omega)A + \Gamma B = -\Gamma G_N. \tag{39b}$$

Thus,

$$A = -\frac{\Gamma(1-G_N)}{N\Gamma + 2\Omega}, \quad B = 1 - \frac{\Omega(1-G_N)}{N\Gamma + 2\Omega},$$

and the stationary particle density is

$$\rho_s = 1 - G_1 = 1 - A - B = \frac{(\Gamma + \Omega)(1-G_N)}{N\Gamma + 2\Omega}. \tag{40}$$

When the initial number of particles is even, $G_N = 1$ and the system gravitates into its absorbing empty state, a state from which it cannot evolve any further. If the initial number of particles is odd ($G_N = 0$), the system can never reach the absorbing state. However, in this case the steady-state density is barely larger than $1/N$ (for N large): the system comes as close to extinction as possible. This generalizes the result of Sudbury that $\rho_s = 0$ for the DBARW in infinite lattices.

Consider now the DBAP, which is the case of no diffusion, $\Gamma = 0$. In this case, Eqs. (39) imply $A = 0$, B arbitrary, or $G_n = \text{const}$. Suppose that the initial number of particles is infinite, so that G_N is not defined (but, since G_n is constant, $0 \leqslant G_N \leqslant 1$). Then, taking the limit $\Gamma \to 0$ in Eq. (40), we find $0 \leqslant \rho_s \leqslant \frac{1}{2}$. This too agrees with Sudbury, who has shown that any homogeneous, random initial distribution leads to $\rho_s = \frac{1}{2}$, but other distributions lead to steady states with $0 \leqslant \rho_s \leqslant \frac{1}{2}$. According to Sudbury, the possible steady states are the borders of the product measures ν_p. Indeed, the border of ν_p has constant G_n: the condition that there be an even number of borders in an n segment is equivalent to having the same states at the edges of the segment in the original ν_p measure. But the probability of this event is $G_n = p^2 + (1-p)^2$, independent of n.

An amusing case of the DBAP is when the initial number of particles is finite. In this case, G_N is known exactly. Thus, if the initial number of particles is odd ($G_N = 0$), then the $\Gamma \to 0$ limit of Eq. (40) yields $\rho_s = \frac{1}{2}$, in agreement with Sudbury. It is instructive to see how this steady state manifests itself in the case of finite rings. An example of $N = 6$ is shown in Appendix A. If the initial number of particles is even ($G_N = 1$), Eq. (40) suggests that $\rho_s = 0$, even for finite N. This is, however, not true, and it can be shown [23,26] that the number of particles remains bounded. (For example, two adjacent particles would propagate as a pair forever, diffusing without change.) In this case, the limit $\Gamma \to 0$ is singular: even the tiniest amount of diffusion would land the system in the absorbing state, but the system can never become empty *without* diffusion.

V. DISCUSSION

In conclusion, the method of intervals enables one to obtain exact results for a large class of diffusion-limited annihilation models in one dimension. The usefulness of this approach stems from the direct consideration of parity conservation in the annihilation process; annihilation conserves the number of particles modulo 2. Defining G_n as the probability that n consecutive sites contain an even number of particles exploits this constraint. In some sense, this method of intervals is a generalization of the method of empty intervals used for coagulation processes [1–5]. A further generalization exists that simultaneously models the q-state Potts model, coagulation, and annihilation [31].

The results of this paper can be summarized as follows. The well-known result $c(t) = c_0 e^{8c_0^2 Dt}\text{erfc}(\sqrt{8c_0^2 Dt})$ for diffusion-limited annihilation with an initial Poisson distribution of particles in the real line was reproduced, and we have obtained an analogous solution for the discrete case of a linear lattice. We have studied the nontrivial steady-state concentration resulting from particle input. In particular, it was shown that a smooth crossover occurs in the steady-state concentration as the separation of input particle pairs increases from zero to infinity; as particle pair input crosses over to single-particle input, the concentration dependence changes from $(R/D)^{1/3}$ to $(R/D)^{1/2}$. Our final example involved annihilation with the symmetric birth reaction $A \to 3A$. The method of intervals allows one to fully examine the kinetics of this reaction scheme in finite ringed lattices. As an example, we derived the exact nature of the steady-state solution for even and odd populations, with or without diffusion.

The method of odd/even intervals introduced here for the case of annihilation could be extended along the same lines as the method of empty intervals used for coagulation [2,3,5,36,37]. This will be the subject of future work. For example, the kinetic phase transition associated with reversible coagulation, $A + A \rightleftharpoons A$ [36], might also have an echo in $A + A \to 0$, $A \to 3A$, at least in the case of finite lattices, when there exists a nonempty steady state. Nonhomogeneous systems could be studied with the addition of one variable by focusing on $G_{nm}(t)$, which is the probability that the interval between sites n and m contains an even number of particles [2]. It might also be possible to obtain multiple-point correlation functions by studying the joint probability that several distinct intervals contain odd/even numbers of particles [3,5].

A natural question to ask is whether the interparticle distribution function (IPDF) for annihilation can be computed by the odd/even interval method, as is the case for coagulation (with the method of empty intervals). In the latter case, the IPDF—the probability that the empty space between two particles is of length x—is given by $p(x,t) = c(t)^{-1}\partial^2 E(x,t)/\partial x^2$, where $E(x,t)$ is the probability that an interval of length x is empty [1]. Unfortunately, $\partial^2 G(x,t)/\partial x^2$ does not convey analogous information. We believe that the full hierarchy of multiple-point correlation functions may be obtained through the odd/even interval method, and since it provides a full description of the system, the IPDF could be obtained as well. While this might

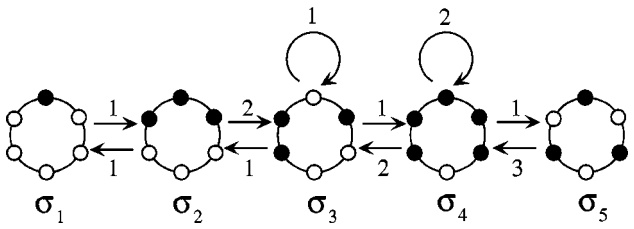

FIG. 4. Configurations of DBAP in an ($N=6$) ring: Empty (solid) circles symbolize empty (occupied) sites. The arrows and numbers indicate relative transition rates between configurations.

work, in principle, the actual computation seems impractical. It would be desirable to find a more straightforward way, based on the odd/even interval method, to compute the IPDF.

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APPENDIX A: STEADY STATE OF DBAP IN $N=6$ RING

Consider the DBAP on a ring of $N=6$ sites, when there is initially an odd number of particles. The system may assume only one of five configurations, σ_i $(i=1,2,\ldots,5)$. These configurations and the transition rates between them are illustrated in Fig. 4. From the figure, we see that the probabilities π_i of having state σ_i obey the rate equations

$$\dot{\pi}_1 = -\pi_1 + \pi_2,$$

$$\dot{\pi}_2 = \pi_1 - 3\pi_2 + \pi_3,$$

TABLE I. G_n for the configurations of DBAP in an ($N=6$) ring, shown in Fig. 4.

	σ_1	σ_2	σ_3	σ_4	σ_5
G_1	5/6	3/6	3/6	1/6	3/6
G_2	4/6	4/6	2/6	4/6	0
G_3	3/6	3/6	3/6	3/6	3/6
G_4	2/6	2/6	4/6	2/6	6/6
G_5	1/6	3/6	3/6	5/6	3/6

$$\dot{\pi}_3 = 2\pi_2 - 2\pi_3 + 2\pi_4,$$

$$\dot{\pi}_4 = \pi_3 - 3\pi_4 + 3\pi_5,$$

$$\dot{\pi}_5 = \pi_4 - 3\pi_5,$$

where dots denote differentiation with respect to time. These equations are linearly dependent, because of the normalization condition $\Sigma_i \pi_i = 1$. Supplementing the rate equations with this condition, we find the steady state: $\pi_1 = \frac{3}{16}$, $\pi_2 = \frac{3}{16}$, $\pi_3 = \frac{6}{16}$, $\pi_4 = \frac{3}{16}$, $\pi_5 = \frac{1}{16}$. The G_n may be computed for each configuration, by averaging over all possible locations of the n segment (Table I). We can now compute the average G_n by weighing the values in Table I with the π_i found above: $\langle G_n \rangle = \Sigma_i \pi_i G_n(\sigma_i)$. This procedure yields the expected result $\langle G_n \rangle = \frac{1}{2}$ $(n=1,2,\ldots,5)$. Rings of other sizes may be analyzed in much the same way, though we found no obvious generalization beyond the simple fact that $\langle G_n \rangle$ is always $\frac{1}{2}$.

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